The Black - Scholes Model

Our 'stochastic base' consists of a probability space, $(\Omega, \mathcal{F}_T, \mathbb{P})$, with a filtration $(\mathcal{F}_t)$, $t \in [0, T]$, which is right continuous, $\mathbb{P}$-complete, $\mathcal{F}_0$ is the trivial σ-field (all $\mathbb{P}$-null sets and all sets of full measure), $\mathcal{F}_t$ is generated by $\mathcal{F}_s$ for $s < t$,

$$\mathcal{F}_t = \sigma\Big(\bigcup_{s<t} \mathcal{F}_s\Big)$$

and $W = (W_t)$ is a BM on this base. On this base live two 'assets', S, a stock and B, a bond.* They satisfy the stochastic equations;*

$$S_t = S_0 + \int_0^t \mu S_s\, ds + \int_0^t \sigma S_s\, dW_s$$

$$B_t = B_0 + \int_0^t r B_s\, ds$$

Exercise : Verify that $S_t = S_0 e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}$

and $B_t = B_0 e^{rt}$.

Here μ, σ, r are positive constants. It is customary to take $B_0 = 1$ or e^{-rT}. We will assume it is 1.

* Or cash account...., we identify S and B with their value.....

A portfolio in S and B is a (stochastic) linear combination of S and B :

$$V_t(\phi, \psi) = \phi_t S_t + \psi_t B_t$$

where (ϕ_t), (ψ_t) are (progressively) measurable adapted processes. There are other technical requirements for ϕ and ψ but we will deprecate these. Roughly speaking; we require ϕ to be integrable with respect to S and ψ integrable with respect to B. The pair (ϕ, ψ) is also called a trading strategy with time t value $V_t(\phi,\psi)$. NB. Everything in sight here is a function of $\omega \in \Omega$.

Portfolios in S and B generate a wide class of 'assets' whose values vary over time and $\omega \in \Omega$ (the states of the world). The discounted value of an asset, (A_t), say, is simply

$$\tilde{A}_t = \frac{A_t}{B_t}$$

So $\tilde{A}_t$ is simply the value of A_t measured in units of the bond B. It turns out that discounted values are very import-

-ant in Mathematical Finance.

For a trading strategy, (ϕ, ψ), the gain from trading in ϕ and ψ is defined to be

$$G_t(\phi, \psi) = \int_0^t \phi_s \, dS_s + \int_0^t \psi_s \, dB_s$$

Note that each of these integrals are understood to be the limit over partitions of belated Riemann Stieltjes sums, i.e. of the form,

$$\sum_{\pi} \phi_{t_{i-1}} \Delta S_{t_i} + \sum_{\pi} \psi_{t_{i-1}} \Delta B_{t_i}$$

where $\Delta S_{t_i} = S_{t_i} - S_{t_{i-1}}$, $\Delta B_{t_i} = B_{t_i} - B_{t_{i-1}}$, π is a partition of $[0, t]$. The essential point is these are stochastic integrals. (Rem: $\neg$ Prescience)

A strategy, (ϕ, ψ), is self-financing if

$$V_t(\phi, \psi) = V_0(\phi, \psi) + G_t(\phi, \psi) .$$

That is, the portfolio value is the inital value plus the gain from trading in S and B with strategies ϕ and ψ. (Rem: $\neg$ Funds added or removed)

Apart from the integrability

conditions alluded to above there are other conditions which we must impose upon the trading strategies. These arise from two possibilities;

(i) Since we allow short selling of S and borrowing of B; i.e., Φ and Ψ can take negative values, we need to limit the extent to which we can pile up debt. Because if we do not then there may be strategies which (almost surely) give a strictly positive payoff. Such a strategy is a riskless way of making money. A concrete example of this kind of thing, albeit in a different mathematical setting, is furnished by "double or nothing betting". Imagine repeated tosses of a fair coin. At each toss one can play a stake of $k \in \mathbb{N}$. If the coin turns up 'heads' you win k and keep your stake. If 'tails' appears you lose your stake. You begin by staking \$1 on the outcome H. If this fails you stake \$2 on H at the next toss. If this fails you stake $\$2^2$ on H at the next toss... and so on. If after n tosses you lost (n times) and at the $(n+1)$th toss a H appears then your winnings are $\$2^n$ while your losses amount to, $\$(2^0 + 2^1 + 2^2 + \dots + 2^{n-1}) = \$(2^n - 1)$. So you've won $1. Now for an unlimited number of tosses there is only one outcome for which you lose at every toss, namely $(T, T, T, \dots, T, \dots)$. For all others you win $1 the first time a H appears. After this you can elect to stop. Allowing an infinite number of tosses and defining a measure on the resulting sequence space from $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$, gives us a strategy which almost surely has a strictly positive payoff so long as we can 'stay in the game' for as long as we wish. This requires very deep pockets, or an obliging bank with unbounded resources!

(ii) One can construct strategies which start with a strictly positive wealth and systematically reduce this towards zero. These so called 'suicide strategies' are also barred from our set of available trading strategies. See Harrison & Pliska for more details.

Accordingly, we impose a further condition on our set of self-financing trading strategies:

Let ξ be a non-negative, integrable, $\mathcal{F}_T$ measurable random variable. We will demand that

$$\tilde{V}_t(\phi, \psi) \geq -M_t^{\mathbb{P}}(\xi)$$

and write $(\phi, \psi) \in SF(\xi)$. Such strategies have discounted values which cannot lose more than an amount $\xi(\omega)$.

A strategy in $SF(\xi)$ will be called admissible. One obvious choice for ξ is $\xi = 0$.

An arbitrage opportunity is a self-financing portfolio, $(V_t(\phi,\psi))$, with the property that

$$V_0(\phi, \psi) \leq 0, \quad V_T(\phi, \psi) \geq 0 \text{ a.s.},$$

$$\mathbb{P}\{\omega : V_T(\phi, \psi)(\omega) > 0\} > 0.$$

Clearly one can replace V with $\tilde{V}$ in the definition above.

We are going to prove some

results now which tie together the ideas of discounting, self-financing and admissibility.

Lemma: Let ϕ, ψ be (progressively) measurable adapted processes satisfying the integrability conditions for S and B respectively. Then (ϕ, ψ) is a self-financing trading strategy if and only if

$$\tilde{V}_t(\phi,\psi) = V_0(\phi,\psi) + \int_0^t \phi_s \, d\tilde{S}_s$$

Pf

If (ϕ,ψ) is self-financing then

$$V_t(\phi,\psi) = V_0(\phi,\psi) + \int_0^t \phi_s \, dS_s + \int_0^t \psi_s \, dB_s$$

Since $\tilde{V}_t(\phi,\psi) = e^{-rt} V_t(\phi,\psi)$ then by the Ito Product Rule:

$$\tilde{V}_t(\phi,\psi) = \int_0^t V_s(\phi,\psi) \, d(e^{-rs}) + \int_0^t e^{-rs} dV_s(\phi,\psi)$$

$$+ \langle V(\phi,\psi), e^{-rs} \rangle + V_0(\phi,\psi)$$

Since (e^{-rs}) is a bounded variation process, the cross-variation term is zero. So,

$$\tilde{V}_t(\phi,\psi) = V_0(\phi,\psi) + \int_0^t -r e^{-rs} V_s(\phi,\psi) \, ds + \int_0^t e^{-rs} \phi_s \, dS_s + \int_0^t e^{-rs} \psi_s \, dB_s.$$

Using the rules, "$dS_s = \mu S_s ds + \sigma S_s dW_s$" and "$dB_s = rB_s ds$" we get

$$\tilde{V}_t(\phi,\psi) = V_0(\phi,\psi) - r\int_0^t \tilde{V}_s(\phi,\psi)\,ds + \int_0^t \phi_s \mu \tilde{S}_s\,ds + \int_0^t \phi_s \sigma \tilde{S}_s\,dW_s + r\int_0^t \psi_s\,ds \quad \ldots\ldots (1)$$

We observe that, using the product rule,

$$\tilde{S}_t = S_0 + \int_0^t S_s\,d(e^{-rs}) + \int_0^t e^{-rs}\,dS_s$$
$$= S_0 + \int_0^t (\mu - r)\tilde{S}_s\,ds + \int_0^t \sigma \tilde{S}_s\,dW_s .$$

Also,

$$\tilde{V}_t(\phi,\psi) = \phi_t \tilde{S}_t + \psi_t , \quad \text{so that}$$
$$\psi_t = \tilde{V}_t(\phi,\psi) - \phi_t \tilde{S}_t \quad \text{—} \quad *$$

and $r\int_0^t \psi_s\,ds = r\int_0^t V_s(\phi,\psi)\,ds - \int_0^r r\phi_s \tilde{S}_s\,ds$

Substituting in (1) above and observ-ing the cancellations we get.

$$\tilde{V}_t(\phi,\psi) = V_0(\phi,\psi) + \int_0^t \phi_s(\mu - r)\tilde{S}_s\,ds + \int_0^t \phi_s \sigma \tilde{S}_s\,dW_s$$
$$= V_0(\phi,\psi) + \int_0^t \phi_s\,d\tilde{S}_s .$$

Remark at this point that $*$

shows that the amount of bonds in a self-financing portfolio is determined by the current discounted portfolio value and the discounted wealth held in S.

For the converse: If we have (partial) strategies, ϕ, ψ, and the portfolio value arising satisfies

$$\tilde{V}_t(\phi,\psi) = V_0(\phi,\psi) + \int_0^t \phi_s \, d\tilde{S}_s$$

then (product rule)

$$V_t(\phi,\psi) = V_0(\phi,\psi) + \int_0^t e^{rs}\phi_s \, d\tilde{S}_s + \int_0^t \tilde{V}_s r e^{rs} ds$$

$$= V_0(\phi,\psi) + \int_0^t e^{rs} \phi_s \, d\tilde{S}_s + \int_0^t r V_s \, ds$$

$$= V_0(\phi,\psi) + \int_0^t \phi_s e^{rs} (\mu - r) \tilde{S}_s \, ds + \int_0^t \phi e^{rs} \sigma \tilde{S}_s \, dW_s$$

$$+ \int_0^t r \phi_s S_s \, ds + \int_0^t r \psi_s B_s \, ds$$

$$= V_0(\phi,\psi) + \int_0^t \phi_s \mu S_s \, ds + \int_0^t \psi_s \, dB_s$$

and (ϕ,ψ) is self-financing.

<u>Lemma</u>

Suppose that under the measure $\mathbb{P}$, $\tilde{S}$ is a martingale and that ϕ is a (partial) strategy such that

$$\left(\int_0^t \phi_s \, d\tilde{S}_s \right)$$

is a local martingale. Define a portfolio value by setting

$$\tilde{V}_t = x_0 + \int_0^t \phi_s \, d\tilde{S}_s \quad , \; x_0 \in \mathbb{R},$$

and a partial strategy ψ by

$$\psi_t = \tilde{V}_t - \phi_t \tilde{S}_t$$

Then (ϕ, ψ) is a self-financing strategy with $\tilde{V}_t \equiv \tilde{V}_t(\phi,\psi)$. If ϕ is chosen so that $\phi \in SF(\xi)$ then $(\tilde{V}_t(\phi,\psi))$ is a supermartingale.

<u>Pf</u>

If (H_t) is a local martingale there is a sequence of stopping times, $\tau_n \uparrow \infty$ such that

$$H^n_t = H_{\tau_n \wedge t}$$

defines an L^2 martingale. Consequently

$$H^n_s = M^{\mathbb{P}}_s(H^n_t) \qquad t \geq s$$

ie. $$H_{\tau_n \wedge s} = M^{\mathbb{P}}_s(H_{\tau_n \wedge t})$$

letting $n \to \infty$ we have

$$\begin{aligned} H_s &= \liminf_n H_{\tau_n \wedge s} \\ &= \liminf_n M^{\mathbb{P}}_s(H_{\tau_n \wedge t}) \end{aligned}$$

$$\overset{(*)}{\geq} M_s^{\mathbb{P}}\left(\liminf_n H_{\tau_n \wedge t}\right)$$
$$= M_s^{\mathbb{P}}(H_t)$$

(*) Uses Fatou's Lemma for conditional expectations. So, (H_t) is a supermartingale.

<u>Lemma</u>

With $\phi, \psi, V_t(\phi,\psi)$ as in the previous lemma, then (ϕ,ψ) cannot be an arbitrage opportunity.

<u>Pf</u> If it were then we would have $V_T(\phi,\psi) \geq 0$ and $x_0 \leq 0$, $P\{\omega : V_T(\phi,\psi)(\omega) > 0\} > 0$. Which precludes $\mathbb{E}(V_T(\phi,\psi)) \leq 0$. However $\tilde{V}_T(\phi,\psi)$ is a supermartingale so that

$$M_0^{\mathbb{P}}(\tilde{V}_T(\phi,\psi)) \leq \tilde{V}_0(\phi,\psi) = x_0$$

So that $\mathbb{E}(\tilde{V}_T(\phi,\psi)) \leq 0$ ※.

Remark: If we take $\xi = 0$ then $x_0 \geq 0$ and $(\tilde{V}_t(\phi,\psi))$ is a non-negative supermartingale.

There is a lot more to say about admissible strategies but we will leave this subject with one last observation: If our strategy (ϕ,ψ) is such that $\tilde{S}$ and

$$\left(\int_0^t \phi_s \, d\tilde{S}_s\right)$$

is a $\mathbb{P}$-martingale then $(\tilde{V}_t(\phi,\psi))$ is a $\mathbb{P}$-martingale too.
Observe also that the random variable $V_T(\phi,\psi)$ is "achieved" by adopting the strategy (ϕ,ψ) in S and B. We say that (ϕ,ψ) replicates $V_T(\phi,\psi)$ or that $V_T(\phi,\psi)$ is attainable via a portfolio in S and B.

Questions:

1. Is $\tilde{S}$ a $\mathbb{P}$ martingale?
2. Are all random variables in $L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ attainable?
3. What is the time $t=0$ value of an attainable claim?

We can begin to answer these questions by looking at the special case, $\tilde{S}$ is a $\mathbb{P}$-martingale. Let X be an $L^2(\mathcal{F}_T)$ random variable which is non-negative. Let (ϕ,ψ) be an admissible strategy that replicates X. So,

$$V_T(\phi,\psi) = X.$$

We also know that

$$\tilde{X} = \tilde{V}_T(\phi,\psi) = V_0(\phi,\psi) + \int_0^T \phi_s d\tilde{S}_s$$

It follows that

$$\mathbb{E}^{\mathbb{P}}(\tilde{X}) = V_0(\phi,\psi) + \mathbb{E}\left(\int_0^T \phi_s d\tilde{S}_s\right)$$

$$= V_0(\phi,\psi),$$

because stochastic integrals with respect to martingales have zero expectation. Now $V_0(\phi,\psi)$ is the initial value of the portfolio given by (ϕ,ψ). Let (ϕ',ψ') be another admissible trading strategy with

$$V_T(\phi',\psi') = X.$$

Then,

$$V_0(\phi,\psi) + \int_0^T \phi \, d\tilde{S}_s = V_0(\phi',\psi') + \int_0^T \phi' d\tilde{S}_s$$

and

$$\int_0^T (\phi - \phi') \, d\tilde{S}_s = V_0(\phi',\psi') - V_0(\phi,\psi).$$

Observe that the left side is a random variable while the right side is a.s. constant (since $\mathcal{F}_0$ is trivial...). Taking $\mathbb{E}^{\mathbb{P}}$ of both sides shows that $V_0(\phi,\psi) = V_0(\phi',\psi')$. Further,

$$\int_0^T (\phi - \phi')\, d\tilde{S}_s = 0$$

therefore the L^2-martingale $\left(\int_0^t (\phi-\phi')\, d\tilde{S}_s\right)$ is zero and therefore

$$\left\langle \int_0^t (\phi - \phi')\, d\tilde{S}_s \right\rangle = \int_0^t (\phi - \phi')^2\, d\langle \tilde{S} \rangle_s$$

$$= 0.$$

So, as long as $(\langle \tilde{S} \rangle_s)$ is a.s. a strictly increasing process, then $(\phi - \phi')$ must be (equivalent to) the zero process. So ϕ and ϕ' are essentially "the same". It follows that ψ and ψ' will be "the same" too. There is, under these circumstances, a unique trading strategy which attains a given random variable in $L^2(\mathcal{F}_T)^+$. (†)

As you may have guessed, the initial value of the portfolio given by (ϕ, ψ) which replicates X must be the time $t=0$ value of X. This begs the question of what exactly do we mean by "the time $t=0$ value of X".

(†) Many of you will see that the technical details ommited here are not entirely trivial.

Item ① in our list of questions has been assumed to have an affirmative answer. Item ③ seems to have the answer $V_0(\phi,\psi)$ where (ϕ,ψ) replicates $\tilde{X}$. For item ② we note that we would need

$$\tilde{X} = V_0(\phi,\psi) + \int_0^T \phi_s d\tilde{S}_s$$

$$= V_0(\phi,\psi) + \int_0^T \phi_s \tilde{S}_s(\mu-r)ds$$

$$+ \int_0^T \phi_s \sigma \tilde{S}_s dW_s$$

or, suggestively,

$$\tilde{X} = V_0(\phi,\psi) + \int_0^T \phi_s \sigma \tilde{S}_s d\left(W_s + \frac{(\mu-r)}{\sigma}s\right).$$

In fact, for $\tilde{S}$ to be a $\mathbb{P}$ martingale we must have $\mu = r$ so that

$$\tilde{X} = V_0(\phi,\psi) + \int_0^T \phi_s \sigma \tilde{S}_s dW_s \quad (2)$$

and our question amounts to asking if for each $X \in L^2(\mathcal{F}_T)$ we can find a ϕ such that (2) is true?

The martingale representation theorem tells us that given $X \in L^2(\mathcal{F}_T)$ there is (η_t) such that

$$\tilde{X} = x_0 + \int_0^T \eta_s dW_s$$

rewriting gives

$$\tilde{X} = x_0 + \int_0^T \frac{\eta_s}{\sigma \tilde{S}_s} \cdot \sigma \tilde{S}_s \, dW_s$$

$$= x_0 + \int_0^T \phi_s \sigma \tilde{S}_s \, dW_s$$

$$= x_0 + \int_0^T \phi_s \, d\tilde{S}_s ,$$

and we know how to make (ϕ, ψ) from this,

$$\psi_t = M_t^{\mathbb{P}}(\tilde{X}) - \phi_t \tilde{S}_t$$

then $\tilde{V}_t(\phi, \psi) = M_t^{\mathbb{P}}(\tilde{X})$ and (ϕ, ψ) replicates X.

As we noted, $\tilde{S}$ is a $\mathbb{P}$-martingale only if $\mu = r$. What do we do if $\mu \neq r$? The answer is provided by Girsanov's Theorem. Return to the equation just prior to (2) above: we would need a (progressively) measurable adapted process, ϕ, such that

$$\tilde{X} = V_0(\phi, \psi) + \int_0^T \phi_s \sigma \, d\left(W_s + \left(\frac{\mu - r}{\sigma}\right) s\right)$$

Now under $\mathbb{P}$ the process $\left(W_t + \left(\frac{\mu - r}{\sigma}\right) t\right)$

is not a Brownian Motion. However by introducing the measure $\mathbb{Q}$ on $\mathcal{F}_T$;

$$\mathbb{Q}(E) = \int_E e^{-\left(\frac{\mu-r}{\sigma}\right)W_T - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T} d\mathbb{P}$$

we know that $\left(W_t + \left(\frac{\mu-r}{\sigma}\right)t\right)$ is a $\mathbb{Q}$-martingale. It may help to write $W_t^* = W_t + \left(\frac{\mu-r}{\sigma}\right)t$ occasionally. Now for an $\mathcal{F}_T$ measurable random variable, X, which lies in $L^2(\Omega, \mathcal{F}_T, \mathbb{Q})^+$ we can apply the martingale representation theorem to write

$$\tilde{X} = x_0 + \int_0^T \eta_s^* \, dW_s^*$$

and rewriting as before....

$$= x_0 + \int_0^T \frac{\eta_s^*}{\sigma \tilde{S}_s} \sigma \tilde{S}_s \, dW_s^* .$$

We further observe, again, that

$$\begin{aligned}
\tilde{S}_t &= S_0 + \int_0^t (\mu - r)\tilde{S}_s \, ds + \int_0^T \sigma \tilde{S}_s \, dW_s \\
&= S_0 + \int_0^t \sigma \tilde{S}_s \, d\left(W_s + \left(\frac{\mu-r}{\sigma}\right)s\right) \\
&= S_0 + \int_0^t \sigma \tilde{S}_s \, dW_s^* .
\end{aligned}$$

So, writing $\phi_s = \dfrac{\eta_s^*}{\sigma \tilde{S}_s}$,

$$\tilde{X} = x_0 + \int_0^T \phi_s \, d\tilde{S}_s$$

where $(\tilde{S}_t)$ is the $\mathbb{Q}$-martingale

$$\tilde{S}_t = S_0 e^{\sigma W_t^* - \frac{\sigma^2}{2} t}.$$

We know already how to form a self-financing portfolio, (ϕ, ψ) which will replicate X.

To summarize: Given our inital assumptions, the is a probability measure, $\mathbb{Q}$, equivalent to $\mathbb{P}$ such than <u>any</u> $L^2(\Omega, \mathcal{F}_T, \mathbb{Q})^+$ random variable is attainable by a self-financing (and admissible) portfolio in S and B. The inital value of the portfolio that replicates X is,

$$x_0 = \mathbb{E}^{\mathbb{Q}}(\tilde{X}).$$

We know also that there are no arbitrage opportunities resulting from admissible portfolio's in S and B. As a consequence one cannot fashion arbitrage from assets, $X, Y \in L^2(\Omega, \mathcal{F}_T, \mathbb{Q})^+$ for each of these is attainable and this would mean an arbitrage from a portfolio in S and B.

Some of you will be asking; What about assets in $L^2(\mathbb{P})^+$ rather than $L^2(\mathbb{Q})^+$? This is a fair point. For an $\mathcal{F}_T$ measurable X to be in $L^2(\Omega, \mathcal{F}_T, \mathbb{Q})$ we need,

$$\mathbb{E}^{\mathbb{P}}\left(e^{-\left(\frac{\mu-r}{\sigma}\right)W_T - \left(\frac{\mu-r}{\sigma}\right)^2 \frac{T}{2}} X^2 \right) < \infty$$

whereas to Y to be in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ we require

$$\mathbb{E}^{\mathbb{P}}(Y^2) < \infty .$$

What is the relationship between $L^2(\mathbb{P})$ and $L^2(\mathbb{Q})$? This is an interesting detour, but a detour nonetheless. I will pass on with the following observation; if $X \in L^p(\mathbb{P})$ for some $p > 2$ then $X \in L^2(\mathbb{Q})$. This allows us to model a vast class of financial instruments.

The Black-Scholes Formula

The original argument follows the following steps.

We let $c(t,x)$ be a $C^{1,2}$ function of the real variables t and x. It is assumed that the time t value

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of a call option on S, struck at K with expiry at time T is given by $C(t, S_t)$ (with some more conditions to follow!). The idea is to find – somehow – a self-financing admissible portfolio arising from (ϕ, ψ) – whatever they are — such that

$$C(t, S_t) = V_t(\phi, \psi)$$

while $\lim_{t \uparrow T} C(t, S_t) = (S_T - K)^+$.

So let us suppose that we have our portfolio, (ϕ, ψ) which is self-financing, so that

$$V_t(\phi, \psi) = V_0(\phi, \psi) + \int_0^t \phi_s dS_s + \int_0^t \psi_s dB_s$$

while, using Itô's Lemma,

$$C(t, S_0) = C(0, S_0) + \int_0^t \frac{\partial C}{\partial t}(s, S_s) ds + \int_0^t \frac{\partial C}{\partial x}(s, S_s) dS_s$$

$$+ \frac{1}{2} \int_0^t \frac{\partial^2 C}{\partial x^2}(s, S_s) \, d\langle S_s \rangle,$$

(why?) and we note that $\langle S_s \rangle = \sigma^2 S_s^2 ds$.

If the portfolio and option value are to agree then their difference must be zero so $C(t,S_t) - V_t(\phi,\psi) = 0$ yeilds

$$C(0,S_0) - V_0(\phi,\psi) + \int_0^t \frac{\partial C}{\partial t}(s,S_s)\,ds + \int_0^t \left(\frac{\partial C}{\partial x}(s,S_s) - \phi_s\right) dS_s$$

$$+ \tfrac{1}{2}\int_0^t \frac{\partial^2 C}{\partial x^2}(s,S_s)\sigma^2 S_s^2\,ds - \int_0^t \psi_s r B_s\,ds$$

$$= 0.$$

By inspection we observe that choosing $\phi_s = \frac{\partial C}{\partial x}(s,S_s)$ zero's the stochastic integral term. This would leave us with,

$$C(0,S_0) - V_0(\phi,\psi) + \int_0^t \left(\frac{\partial C}{\partial t}(s,S_s) + \frac{1}{2}\frac{\partial^2 C}{\partial x^2}(s,S_s)\sigma^2 S_s^2 - \psi_s r B_s\right) ds.$$

Obviously we want $C(0,S_0) - V_0(\phi,\psi) = 0$ but at this stage we have no way of determining on via the other. However, for every $0 < t \leq T$ we would want

$$0 = \int_0^t \left(\frac{\partial C}{\partial t}(s,S_s) + \frac{1}{2}\frac{\partial^2 C}{\partial x^2}(s,S_s)\sigma^2 S_s^2 - \psi_s r B_s\right) ds$$

We can achieve this by setting

$$\psi_s = \frac{\frac{\partial C}{\partial t}(s,S_s) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(s,S_s)\sigma^2 S_s^2}{r B_s}$$

This done we would have, for $t > 0$,

$$C(t,S_t) = \phi_t S_t + \psi_t B_t$$

$$= \frac{\partial C}{\partial x}(t,S_t) S_t + \frac{\left(\frac{\partial C}{\partial t}(t,S_t) + \frac{1}{2}\frac{\partial^2 C}{\partial x^2}(t,S_t)\,\sigma^2 S_t^2\right)}{r}$$

Missing r!

or, rearranging, for $t > 0$,

B&S r pde

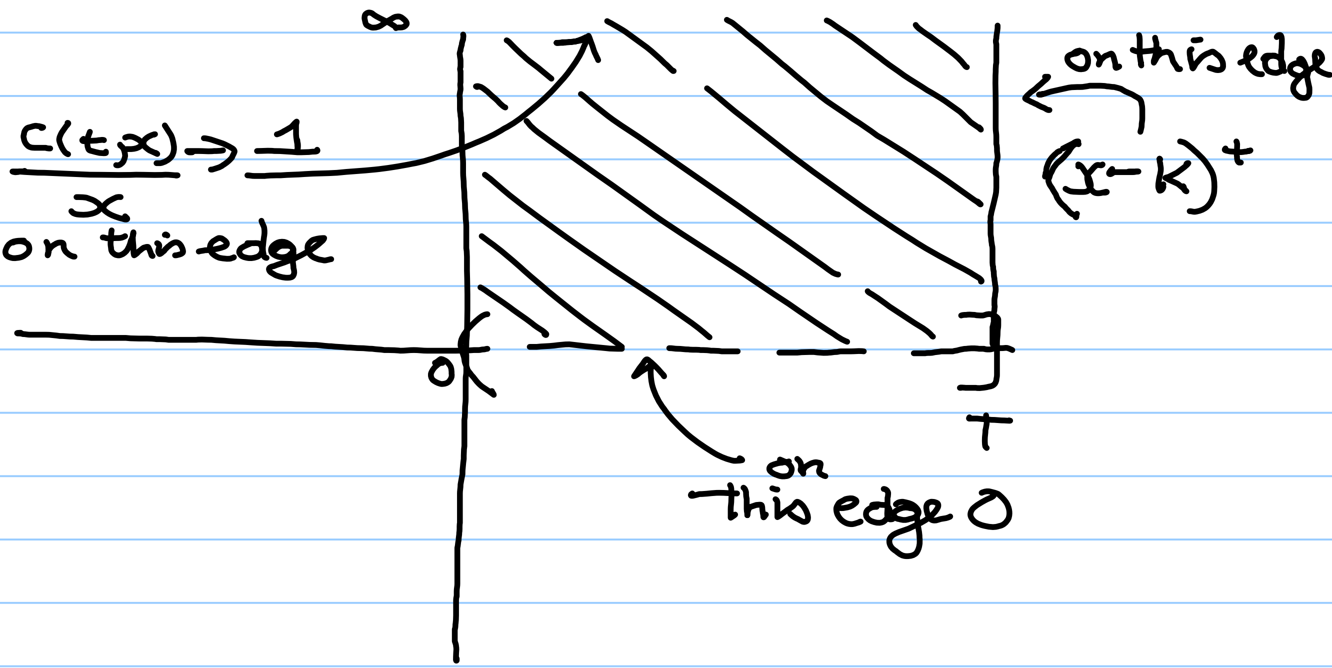
$$\frac{\partial C}{\partial x}(t,S_t) S_t + \frac{\partial C}{\partial t}(t,S_t) + \frac{1}{2}\frac{\partial^2 C}{\partial x^2}(t,S_t)\,\sigma^2 S_t^2 - rC(t,S_t) = 0.$$

Now, as t and ω vary $S_t(\omega)$ takes all values in $(0,\infty)$ showing that $C(t,x)$ satisfies the partial differential equation

B&S r pde

$$x\frac{\partial C}{\partial x} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} - rC = 0$$

in the region $(0,T] \times (0,\infty)$.



Note that, up to now, we have not used the fact that we require $C(T, S_T) = (S_T - K)^+$. Indeed, we could have prosecuted this argument for any (suitable) function, $f(S_T)$. All of these "options" would have values that satisfied the pde above in the region $(0,T] \times (0,\infty)$. To obtain a specific solution to this pde in the region $[0,T] \times [0,\infty)$ we need boundary conditions. It is at this point that specific features of the option contract enter the picture. So, for example, on the right edge of our region we require $C(T, x) = (x-K)^+$. As we move toward the upper edge, where "x tends to infinity" it is reasonable to assume that $C(T, x)/x$ tends to 1, while as x tends to 0 $C(t, x)$ should tend to 0 also. These three boundary conditions determine the fourth! That is there is only one solution satisfying the pde with these boundary conditions. So along the $t=0$ edge our solution will tell us the time zero value $C(0, S_0)$.

But we run ahead of ourselves. Is our portfolio actually self-financing? We 'know' that

$$C(t,S_t) = \phi_t S_t + \psi_t B_t$$

while

$$\int_0^t \phi_s \, dS_s + \int_0^t \psi_s \, dB_s = \int_0^t \frac{\partial C}{\partial x}(s,S_s)\, dS_s + \int_0^t \left(\frac{\frac{\partial C}{\partial t}(s,S_s) + \frac{1}{2}\sigma^2 S_s^2 \frac{\partial^2 C}{\partial x^2}(s,S_s)}{r B_s} \right) dB_s$$

$$= \int_0^t \frac{\partial C}{\partial x}(s,S_s)\, dS_s + \int_0^t \frac{\partial C}{\partial t}(s,S_s)\, ds + \int_0^t \frac{1}{2}\sigma^2 S_s^2 \frac{\partial^2 C}{\partial x^2}(s,S_s)\, ds$$

$$= C(t,S_t) - C(0,S_0)$$

So $\phi_t S_t + \psi_t B_t = C(0,S_0) + \int_0^t \phi_s \, dS_s + \int_0^t \psi_s \, dB_s$

and the portfolio is self-financing as long as its inital amount is $C(0,S_0)$.

You may find this argument less than satisfying! There was a minor industry devoted to getting this approach correct and consistent. However, the argument provided a key insight that informed many aspects of Mathematical Finance and alternative approaches to pricing securities. See the exercises for some calculations for solving the B+S p.d.e.

Before we move on let us reiterate something: If a 'payoff', $X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, can be replicated by an admissible (self-financing) portfolio then the time $t=0$ value of X must coincide with the time $t=0$ value of the portfolio; writing X_0 for the time $t=0$ value of X and supposing $X_0 > V_0(\phi,\psi)$ then we "sell" X at $t=0$ and "buy" (ϕ,ψ), for $V_0(\phi,\psi)$. This portfolio, which is short X and long (ϕ,ψ), is managed until time T when it's assets amount to $V_T(\phi,\psi) - X + (X_0 - V_0(\phi,\psi))e^{rT}$, since we put the spare cash in the bond. Since $V_T(\phi,\psi) = X$ this portfolio, which has zero initial endowment, is an arbitrage and moreover it is an arbitrage in S and B because X is simply the result of (ϕ,ψ). Effectively, what this shows, with the argument completed for the case $X_0 < V_0(\phi,\psi)$, is that the time $t=0$ value of (ϕ,ψ) is unique. As we saw before, given $X \in L^2(\Omega)^+$ the strategy (ϕ,ψ) is "essentially" unique and so too is it's initial value.

Exercise : Write a formal mathematical proof that X_0 must be $V_0(\phi, \psi)$.

Pricing the Call Option by Martingale Techniques.

Our call option has payoff $(S_T - K)^+$ and our previous work shows that the time $t = 0$ value of the call option is

$$C_0 = \mathbb{E}^{\mathbb{Q}}\left(\frac{(S_T - K)^+}{e^{rT}} \right).$$

Here $\mathbb{Q}$ is the measure equivalent to $\mathbb{P}$ given by

$$\mathbb{Q}(E) = \int_E e^{-\left(\frac{\mu - \sigma}{r}\right) W_T - \left(\frac{\mu - \sigma}{r}\right)^2 T/2} \, d\mathbb{P}$$

For which $\left(W_t + \left(\frac{\mu - \sigma}{r}\right) t \right)$ is a $\mathbb{Q}$ Brownian Motion. Now under $\mathbb{P}$,

$$\begin{aligned} \tilde{S}_t &= S_0 e^{\sigma W_t + (\mu - r - \sigma^2/2)t} \\ &= S_0 e^{\sigma\left(W_t + \left(\frac{\mu - r}{\sigma}\right) t \right) - \sigma^2/2\, t} \\ &= S_0 e^{\sigma W_t^* - \sigma^2/2\, t} \end{aligned}$$

writing $W_t^* = W_t + \frac{(\mu - r)}{\sigma} t$. So under $\mathbb{Q}$, $\tilde{S}$, has a different form.

Exercise: Use Itô's Lemma to prove that,

$$\tilde{S}_t = S_0 + \int_0^t \tilde{S}_s \sigma \, dW_s^* \text{, a martingale,}$$

and

$$S_t = S_0 + \int_0^t S_s r \, ds + \int_0^t S_s \sigma \, dW_s^*$$

$$= S_0 e^{\sigma W_t^* + (r - \sigma^2/2)t}$$

So that, under $\mathbb{Q}$, S has drift, r.

Return to $\mathbb{E}^{\mathbb{Q}}\left(\frac{(S_T - K)^+}{e^{rT}}\right)$;

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{(S_T - K)^+}{e^{rT}}\right) = \mathbb{E}^{\mathbb{Q}}\left((S_T - K) I_{\{S_T > K\}} e^{-rT}\right)$$

$$= \mathbb{E}^{\mathbb{Q}}\left(S_T I_{\{S_T > K\}} e^{-rT}\right) - \frac{K}{e^{rT}} \mathbb{E}^{\mathbb{Q}}\left(I_{\{S_T > K\}}\right).$$

We deal with the term involving K first of all.

$$\mathbb{E}^{\mathbb{Q}}\left(I_{\{S_T > K\}}\right) = \mathbb{Q}\{S_T > K\}$$

$$= \mathbb{Q}\left\{S_0 e^{\sigma W_T^* + (r - \sigma^2/2)T} > K\right\}$$

$$= \mathbb{Q}\left\{ e^{\sigma W_T^* + (r - \sigma^2/2)T} > \frac{K}{S_0} \right\}$$

$$= \mathbb{Q}\left\{ \sigma W_T^* > \log\left(\frac{K}{S_0}\right) - (r - \sigma^2/2)T \right\}$$

$$= \mathbb{Q}\left\{ \sigma\sqrt{T}\, N(0,1) > \log\left(\frac{K}{S_0}\right) - (r - \sigma^2/2)T \right\}$$

$$= \mathbb{Q}\left\{ N(0,1) > \frac{\log\left(\frac{K}{S_0}\right) - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \right\}$$

$N(0,1)$ here indicates a standard normal random variable for the measure $\mathbb{Q}$!

$$= \mathbb{Q}\left\{ N(0,1) < \frac{\log\left(\frac{S_0}{K}\right) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \right\}$$

$$= N\left(\frac{\log\left(\frac{S_0}{K}\right) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \right)$$

Where $N(x)$ is the standard normal distribution function. This takes care of the item involving K.

Now we need to evaluate

$$\mathbb{E}^{\mathbb{Q}}\left(S_T I_{\{S_T > K\}} e^{-rT} \right).$$

At first sight this seems difficult.

But, recalling that $S_T = S_0 e^{\sigma W_T^* + (r - \sigma^2/2)T}$

we see that $S_T e^{-rT} = S_0 e^{\sigma W_T^* - \sigma^2/2 T}$.
So we must evaluate

$$\mathbb{E}^{\mathbb{Q}}\left(S_0 e^{\sigma W_T^* - \sigma^2/2 T} I_{\{S_T > K\}}\right).$$

The term S_0 is constant so we can take it out (and ignore it for a bit). It might take a bit of time, but one realises that the exponential term provides a change of measure!
Let

$$\mathbb{R}(E) = \int_E e^{\sigma W_T^* - \sigma^2/2 T} \, d\mathbb{Q},$$

So that,

$$\mathbb{E}^{\mathbb{Q}}\left(S_0 e^{\sigma W_T^* - \sigma^2/2 T} I_{\{S_T > K\}}\right) = S_0 \mathbb{R}(\{S_T > K\})$$

Now, to work out $\mathbb{R}\{S_T > K\}$ we need to know the equation for S_T under the measure $\mathbb{R}$. Recall that Girsanov tells us that $(W_t^* - \sigma t)$ is an $\mathbb{R}$ Brownian Motion. We know,

$$S_T = S_0 e^{\sigma W_T^* + (r - \sigma^2/2)T}$$
$$= S_0 e^{\sigma W_T^* - \sigma^2 T + \sigma^2 T + (r - \sigma^2/2)T}$$

$$= S_0 e^{\sigma(W_T^* - \sigma T) + (r + \sigma^2/2)T}$$

$$= S_0 e^{\sigma \hat{W}_T + (r + \sigma^2/2)T}$$

where $(\hat{W}_t)$ is an $\mathbb{R}$-Brownian Motion. S_0

$$S_0 \mathbb{E}^{\mathbb{Q}}\left(S_T I_{\{S_T > K\}} e^{-rT}\right) = S_0 \mathbb{R}\{S_T > K\}$$

where $S_T = S_0 e^{\sigma \hat{W}_T + (r + \sigma^2/2)T}$.

But we have, already, effectively done this calculation. The answer is,

$$S_0 N\left(\frac{\log\left(\frac{S_0}{K}\right) + (r + \sigma^2/2)T}{\sigma \sqrt{T}} \right)$$

Putting the two parts together gives the classic option pricing formula.

Put - Call Parity

One could recapitulate these arguments for the Put Option, whose payoff is $(K - S_T)^+$. But there is a nice piece of algebra that helps:

$$(S_T - K)^+ - (K - S_T)^+$$

$$= (S_T - K) I_{\{S_T \geq K\}} - (K - S_T) I_{\{S_T < K\}}$$

$$= S_T I_{\{S_T \geq K\}} + S_T I_{\{S_T < K\}}$$

$$- K I_{\{S_T \geq K\}} - K I_{\{S_T < K\}}$$

$$= S_T - K .$$

Consequently

$$\mathbb{E}^{\mathbb{Q}}\left((S_T - K)^+ e^{-rT} \right) - \mathbb{E}^{\mathbb{Q}}\left((K - S_T)^+ e^{-rT} \right)$$

$$= \mathbb{E}^{\mathbb{Q}}\left(S_T e^{-rT} \right) - \mathbb{E}^{\mathbb{Q}}\left(K e^{-rT} \right) .$$

Write C_0, P_0, for the $t = 0$ values of the call and put we can rewrite this as

$$C_0 - P_0 = S_0 - \frac{K}{e^{rT}} .$$

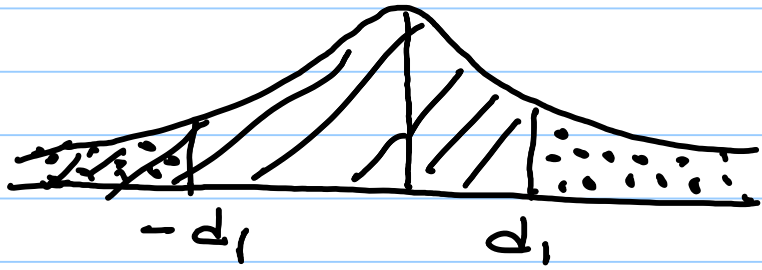
We have just calculated C_0 and this relation allows us to calculate P_0. Of course,

$$P_0 = C_0 - S_0 + \frac{K}{e^{rT}}$$

$$= S_0 N(d_1) - K e^{-rT} N(d_2) - S_0 + K e^{-rT}$$

$$= S_0(N(d_1)-1) + Ke^{-rT}(1-N(d_2))$$
$$= Ke^{-rT}(1-N(d_2)) - S_0(1-N(d_1)) .$$

Now



/// $= N(d_1)$
$\because$ $= 1 - N(d_1)$
$= N(-d_1)$

So,
$$P_0 = Ke^{-rT}N(-d_2) - S_0N(-d_1) .$$

Exercise : Can you devise an arbitrage argument to derive Call-Put parity? Think of making a portfolio from S, B and possibly C and/or P..... .

<u>Remark</u> Call-Put Parity is a relation that is independent of our model for S so long as we are in a 'no-arbitrage' world.

Options on Dividend Paying Assets.

The theory developed so far applies itself to assets, S, which do not pay dividends! Dividends are a curious feature of stocks and there are differing treatments of dividends in the literature. We will assume that we have a stock, S, which pays dividends at a constant rate, δ. By this we mean that over the time interval $[t, t+dt]$ — where dt is 'small' — a unit of stock pays $\delta S_t\, dt$ as dividend. If we hold a self-financing portfolio in S and B, (ϕ, ψ) say, then the portfolio value, X_t, must be equal to $\phi_t S_t + \psi_t B_t$ but the gain has to comprise four terms, X_0 the initial value, $\int_0^t \phi_s dS_s$, the capital gain or loss due to trading ϕ in S, $\int_0^t \psi_s dB_s$, the gain or loss due to trading ψ in B and, finally, the accumulated value of the dividend stream,

$$\int_0^t \delta \phi_s S_s \, ds .$$

So

$$X_t = X_0 + \int_0^t \phi_s dS_s + \int_0^t \psi_s dB_s + \int_0^t \delta \phi_s S_s \, ds$$

and

$$X_t = \phi_t S_t + \psi_t B_t .$$

Notice that we have not attempted to aportion the accumulated dividend to a part in S and a part in B. We are implicitly assuming that wherever it lies it has not been annihilated by the strategy (ϕ, ψ). This is a very unsatisfactory aspect of this model! Note also that according to our equations above the strategy $(1, 0)$ with initial endowment S_0 shows that

$$X_t = \phi_t S_t + \psi_t B_t = 1 \cdot S_t + 0 \cdot B_t$$
$$= S_t$$

while

$$S_t = S_0 + \int_0^t 1 \, dS_s + \int_0^t 0 \cdot dB_s + \int_0^t \delta S_s \, ds$$
$$= S_0 + S_t - S_0 + \delta \int_0^t S_s \, ds$$

i.e.

$$S_t = S_t + \delta \int_0^t S_s \, ds$$

Which is contrary to the common view that "dividends depress the value of a stock".

With a self-financing portfolio

$$X_t = V_t(\phi, \psi) = \phi_t S_t + \psi_t B_t$$

and

$$X_t = X_0 + \int_0^t \phi_s dB_s + \int_0^t \psi_s dB_s$$

Now $\psi_s = \dfrac{X_s - \phi_s S_s}{B_t}$ so we can rewrite;

$$X_t = X_0 + \int_0^t \phi_s dB_s + \int_0^t (X_s - \phi_s S_s) r \, ds .$$

Suppose now that we have a portfolio whose value at time t is X_t and whose holding is S at time t is ϕ_t with the rest of the wealth invested in the bond. So, the wealth in S is $\phi_t S_t$ and the wealth in B is $X_t - \phi_t S_t$ or put another way

$\psi_t = \dfrac{X_t - \phi_t S_t}{B_t}$. So (ϕ, ψ) defines our holdings throughout $[0, T]$.

Will such a portfolio necessarily be self-financing? Imagine making a portfolio by starting with an initial endowment of

S_0 with which we but 1 unit of S at time $t=0$. We suppose that S pays a continuous dividend at a (constant) rate, δ. So the wealth arising from the dividend stream amounts to

$$\int_0^t \delta S_s \, ds$$

at time t. Let us suppose that the dividend stream is directed into the bond. So, in the interval "$[s, s+ds]$", an amount of wealth "$\delta S_s ds$" arrives and is immediately invested in the bond; it buys

$$\delta \frac{S_s}{B_s} ds$$

bonds. Over time the number of bonds acquired is

$$\int_0^t \delta \tilde{S}_s \, ds$$

and the portfolio value is

$$X_t = S_t + \left(\int_0^t \delta \tilde{S}_s \, ds\right) B_t$$

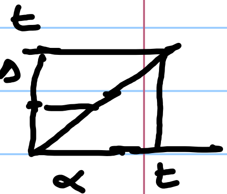
So, clearly, the wealth in bonds is $X_t - \phi_t S_t$ with $\phi_t \equiv 1$ but this portfolio is not self-financing;

Since $\phi_t \equiv 1$, $\int_0^t \phi_s \, dS_s = \int_0^t dS_s = S_t - S_0$

while $\psi_t = \int_0^t \delta \tilde{S}_s \, ds$ so that

$$\int_0^t \psi_s \, dB_s = \int_0^t \left(\int_0^s \delta \tilde{S}_\alpha \, d\alpha \right) dB_s$$

$$= \int_0^t \int_0^s \delta \tilde{S}_\alpha \, d\alpha \; r B_s \, ds$$



$$= \int_0^t \int_0^s \delta r S_\alpha e^{r(s-\alpha)} \, d\alpha \, ds$$

$$= \int_0^t \int_0^\alpha \delta r S_\alpha e^{r(s-\alpha)} \, ds \, d\alpha$$

$$= \int_0^t \delta r S_\alpha \left(\frac{1 - e^{-r\alpha}}{r} \right) d\alpha$$

$$= \int_0^t \delta S_\alpha \, d\alpha - \int_0^t \delta \tilde{S}_\alpha \, d\alpha$$

So

$$X_0 + \int_0^t \phi_s \, dS_s + \int_0^t \psi_s \, dB_s =$$

$$S_0 + S_t - S_0 + \int_0^t \delta S_\alpha \, d\alpha - \int_0^t \delta \tilde{S}_\alpha \, d\alpha$$

$$= S_t + \int_0^t \delta S_\alpha \, d\alpha - \int_0^t \delta \tilde{S}_\alpha \, d\alpha$$

If this is to be X_t we must have

$$S_t + \int_0^t \delta S_\alpha d\alpha - \int_0^t \delta \tilde{S}_\alpha d\alpha = S_t + \left(\int_0^t \delta \tilde{S}_\alpha d\alpha\right) B_t$$

Or

$$\int_0^t \delta S_\alpha d\alpha = \left(\int_0^t \delta \tilde{S}_\alpha d\alpha\right)(1 + B_t)$$

$$\left(\int_0^t \delta \tilde{S}_\alpha d\alpha\right)(1 + B_t) = \int_0^t (1+B_\alpha)\,\delta \tilde{S}_\alpha d\alpha + \int_0^t \left(\int_0^s \delta \tilde{S}_\alpha d\alpha\right) dB_s$$

$$= \int_0^t \delta \tilde{S}_\alpha d\alpha + \int_0^t \delta S_\alpha d\alpha + \int_0^t \delta S_\alpha d\alpha - \int_0^t \delta \tilde{S}_\alpha d\alpha$$

So $\int_0^t \delta S_\alpha d\alpha = 2 \int_0^t \delta S_\alpha d\alpha$ and $1 = 0$.

The problem here seems to be the meaning of "self-financing".

We now cover what I will call a standard treatment of options on a dividend paying asset. You can find this approach in the literature but there are others. They give different answers for the value of options.

We are going to take two slightly different approaches. We will assume two distinct dynamics for S and for each of these make an argument for the portfolio value.

First of all we assume that

$$S_t = S_0 + \int_0^t \mu S_s\,ds + \int_0^t \sigma S_s\,dW_s$$

a portfolio in S and B will have value

$$X_t = X_0 + \int_0^t \phi_s\,dS_s + \int_0^t \frac{(X_s - \phi_s S_s)}{B_s}\,dB_s + \int_0^t \delta \phi_s S_s\,ds$$

where the last term accounts for the accumulated weath due to dividends.

So,

$$X_t = X_0 + \int_0^t \mu \phi_s S_s\,ds + \int_0^t \sigma \phi_s S_s\,dW_s + \int_0^t (X_s - \phi_s S_s) r\,ds + \int_0^t \delta \phi_s S_s\,ds$$

$$= X_0 + \int_0^t (\mu + \delta - r)\phi_s S_s\,ds + \int_0^t \sigma \phi_s S_s\,dW_s + \int_0^t r X_s\,ds$$

$$= X_0 + \int_0^t \phi_s S_s \sigma\, d\left(W_s + \left(\frac{\mu+\delta-r}{\sigma}\right)s\right) + \int_0^t r X_s\,ds$$

Using Girsanov,

$$\mathbb{Q}(E) = \int_E e^{-\left(\frac{\mu+\delta-r}{\sigma}\right)W_T - \frac{1}{2}\left(\frac{\mu+\delta-r}{\sigma}\right)^2 T}\,d\mathbb{P}$$

sees $W_t + \left(\frac{\mu+\delta-r}{\sigma}\right)t$ as Brownian Motion, which we call W^*.

So now, under $\mathbb{Q}$,

$$X_t = X_0 + \int_0^t r X_s\,ds + \int_0^t \phi_s S_s \sigma\,dW_s^*, \text{ and}$$

$$\tilde{X}_t = X_0 + \int_0^t \phi_s \tilde{S}_s \sigma\,dW_s^* \quad \text{(Product Rule)}.$$

We note that under $\mathbb{Q}$,

$$S_t = S_0 + \int_0^t \mu S_s\,ds + \int_0^t \sigma S_s\,d\left(W_s^* - \left(\frac{\mu+\delta-r}{\sigma}\right)s\right)$$

$$= S_0 + \int_0^t (r-\delta) S_s + \int_0^t \sigma S_s\,dW_s^*.$$

Now we do exactly the same calculat-
-ions with different dynamics for S:
We suppose that

$$S_t = S_0 + \int_0^t \mu S_s\, ds + \int_0^t \sigma S_s\, dW_s - \int_0^t \delta S_s\, ds.$$

So the dividend stream "depresses" the value of S. We form a portfolio in S and B and because we "own" it then we receive the dividend stream: So,

$$X_t = X_0 + \int_0^t \phi_s\, dS_s + \int_0^t (X_s - \phi_s S_s) r\, ds + \int_0^t \delta \phi_s S_s\, ds.$$

$$= X_0 + \int_0^t \phi_s S_s (\mu - r)\, ds + \int_0^t \sigma \phi_s S_s\, dW_s + \int_0^t r X_s\, ds.$$

$$= X_0 + \int_0^t \sigma \phi_s S_s\, d\left(W_s + \left(\frac{\mu - r}{\sigma}\right)s\right) + \int_0^t r X_s\, ds$$

Using Girsanov,

$$\hat{\mathbb{P}}(E) = \int_E e^{-\left(\frac{\mu-r}{\sigma}\right)W_T - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T}\, d\mathbb{P}$$

sees $W_t + \left(\frac{\mu-r}{\sigma}\right)t$ as a Brownian Motion which we call $\hat{W}$.
We see that under $\hat{\mathbb{P}}$

$$X_t = X_0 + \int_0^t r X_s\,ds + \int_0^t \phi_s S_s\,d\hat{W}_s$$

so that

$$\tilde{X}_t = X_0 + \int_0^t \phi_s \tilde{S}_s\,d\hat{W}_s .$$

The dynamics of S under $\hat{\mathbb{P}}$ are

$$S_t = S_0 + \int_0^t \mu S_s\,ds + \int_0^t \sigma S_s\,d\left(\hat{W}_s - \left(\frac{\mu - r}{\sigma}\right)s\right) - \int_0^t \delta S_s\,ds$$

$$= S_0 + \int_0^t (r-\delta) S_s\,ds + \int_0^t \sigma S_s\,d\hat{W}_s .$$

If you look back you will see that the dynamics of X and S look the same, indeed in a real sense are the same, from each point of view. In particular $(\tilde{X}_t)$ is a martingale and is the solution of the same stochastic equation in each case.

Each approach is marred by being completely vague about what happens to the dividend stream. But if we set aside our misgivings we can now evaluate options on these kind of assets and make some further observations.

First; in each case, $\mathbb{Q}$ or $\hat{\mathbb{P}}$

$$S_t = S_0 + \int_0^t (r-d) S_s\,ds + \int_0^t \sigma S_s\,dB_s$$

where B is a Brownian Motion. Therefore,

$$\tilde{S}_t = S_0 - \int_0^t \delta \tilde{S}_s\, ds + \int_0^t \sigma \tilde{S}_s\, dB_s$$

So $(\tilde{S}_t)$ is not a ($\mathbb{Q}$ or $\hat{\mathbb{P}}$) martingale. Moreover, the rate of return of S (under $\mathbb{Q}$, or $\hat{\mathbb{P}}$) is not r. However, (for both $\mathbb{Q}$, and $\hat{\mathbb{P}}$) $\tilde{X}$ is a martingale. This allows us to price, for example, a call option on S. We work through the case for a measure $\mathbb{Q}'$, for which B is a Brownian motion as

$$\tilde{X}_t = X_0 + \int_0^t \sigma \phi_s S_s\, dB_s$$

So the discounted portfolio value is a $\mathbb{Q}'$-martingale, that is,

$$\frac{X_t}{e^{rt}} = M_t^{\mathbb{Q}'}\left(\frac{X_T}{e^{rT}}\right).$$

We can ask, given a payoff $Y \in L^2(\mathbb{Q}')$, can we replicate it with a portfolio, (ϕ, ψ), in S and B while S pays a constant dividend at rate δ? The answer via the representation theorem for $\mathbb{Q}'$ martingales is affirmative. We form the martingale $M_t^{\mathbb{Q}'}\left(\frac{Y}{e^{rT}}\right)$. Using MRThm, initial endowment

$$M_t^{\mathbb{Q}'}\left(\frac{Y}{e^{rT}}\right) = \nu_0 + \int_0^t \eta_s \, dB_s$$

$$= \nu_0 + \int_0^t \frac{\eta_s}{\sigma \tilde{S}_s} \sigma \tilde{S}_s \, dB_s$$

$$= \nu_0 + \int_0^t \phi_s \tilde{S}_s \, dB_s$$

where $\phi_s = \dfrac{\eta_s}{\sigma \tilde{S}_s}$ and we define

$$\psi_s = \frac{M_s^{\mathbb{Q}'}\left(\frac{Y}{e^{rT}}\right) - \phi_s S_s}{B_s}.$$

So ν_0 is just $\mathbb{E}^{\mathbb{Q}'}\left(\frac{Y}{e^{rT}}\right)$ as usual and, apparently, this situation is no different from what we have observed before. However, there is a difference, The dynamics of S!

$$S_t = S_0 + \int_0^t (r-\delta) S_s \, ds + \int_0^t \sigma S_s \, dB_s$$

When $Y = (S_T - K)^+$ we know that the time $t = 0$ price will be

$$\mathbb{E}^{\mathbb{Q}'}\left(\frac{(S_T - K)^+}{e^{rT}} \right)$$

as before we can write

$$\mathbb{E}^{\mathbb{Q}'}\left((S_T - K)^+ / e^{rT} \right) = \mathbb{E}\left(\frac{S_T I_{\{S_T > K\}}}{e^{rT}} \right) - \frac{K}{e^{rT}} \mathbb{E}^{\mathbb{Q}'}\left(I_{\{S_T > K\}} \right)$$

This is the same calculation done previously! The difference is that the rate of return of S is $r - \delta$ rather than r. The effect of this is to modify $\mathbb{Q}'\{S_T > K\}$ to be

$$N\left(\frac{\log\left(\frac{S_0}{K}\right) + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}} \right)$$

while the $\mathbb{E}^{\mathbb{Q}'}\left(S_T e^{-rT} I_{\{S_T > K\}} \right)$

is modified to

$$S_0 e^{-\delta T} N\left(\frac{\log\left(\frac{S_0}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).$$

You can get a little more from this situation. If one specifies that there is no bond holding then

$$X_s = \phi_s S_s \quad \text{and} \quad \tilde{X}_s = \phi_s \tilde{S}_s$$

so that,

$$\tilde{X}_t = X_0 + \int_0^t \sigma \tilde{X}_s \, dB_s$$

We can solve this!

$$\tilde{X}_t = X_0 e^{\sigma B_t - \frac{\sigma^2}{2}t}$$

or

$$X_t = X_0 e^{\sigma B_t + \left(r - \frac{\sigma^2}{2}\right)t}$$

If we set $t = T$ and $X_0 = S_0 e^{-\delta T}$ then

$$X_T = S_0 e^{\sigma B_T + \left(r - \delta - \frac{\sigma^2}{2}\right)T}$$

$$\equiv S_T$$

So this shows that the time $t = 0$ price, X_0, of a portfolio that delivers 1 unit of S at time T is $S_0 e^{-\delta T}$. So the forward price of S at time T is $S_0 e^{(r-\delta)T}$. To hedge buy $S_0 e^{-\delta T}$ units of S and reinvest all dividends into S carrying no bonds.